VI. Relaxed Models

- Motivation
- $\varepsilon$-Admissible Speedup Versions of A*
- Using Information about Uncertainty of $h$
- Risk Measures
- Nonadditive Evaluation Functions
- Heuristics Provided by Simplified Models
- Mechanical Generation of Admissible Heuristics
- Probability-Based Heuristics
Using Information about Uncertainty of $h$

Using a non-admissible heuristic function

Idea [Harris]:

The heuristic function $h$ estimates the cheapest remaining cost $h^*$ mostly quite well, but sometimes overestimates $h^*$ by no more than $\varepsilon$.

$\Rightarrow$ A* using such a heuristic function $h$ is $\varepsilon$-admissible.

The condition for $\varepsilon$-admissibility of A* is satisfied because at termination it holds

$$ h(n) - h^*(n) \leq \varepsilon \text{ für alle } n \in \text{OPEN}. $$

Also the weakened form of admissibility of $h$ is often too restrictive.

Often it is easier to find a heuristic estimate for $h^*$ that mostly estimates precisely but sometimes overestimates $h^*$ (by much more than any reasonable $\varepsilon$).

$\Rightarrow$ The error in the estimate is not limited, but a large error is unlikely.
Remarks:

- Heuristic functions $h$ with $h \leq (1 + \varepsilon)h^*$ are called $\varepsilon$-admissible.

- Analogously to Lemma $C^*$-Bounded OPEN Node, it can be proven that, at any point in time before termination, there exists some node $n$ in OPEN with $f(n) \leq (1 + \varepsilon)C^*$.

- The condition "$h(n) - h^*(n) \leq \varepsilon$ for all $n \in OPEN$" is sufficient, but not necessary, for A* being $\varepsilon$-admissible.
Using Information about Uncertainty of $h$

Illustration of Underestimating and Overestimating Estimation Functions
Using Information about Uncertainty of $h$

Example: Search in “Random” Graphs

Given is a graph with randomly drawn edge costs. The minimum number of edges to a target node is known in each node.

- Edge costs $c(n, n')$ are known to be drawn independently from a common distribution function, uniform in interval $[0; 1]$.

- For long paths with $N$ edges from a node $n$ to a goal node in $\Gamma$ it is known that $h^*(n)$ is most likely to be near $\frac{N}{2}$.

- The only *admissible* heuristic estimate for $h^*$ is $h_1(n) = 0$.

- The most reasonable heuristic estimate for $h^*$ is $h_2(n) = \frac{N}{2}$.

The heuristic estimate $h_2$ leads to a worst-case cost overestimation of $\frac{N}{2}$ and is therefore not $(\varepsilon)$-admissible. But the likelihood of this event is extremely small.
Using Information about Uncertainty of $h$

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- The only admissible heuristic estimate for $h^*$ is $h_1(n) = 0$.
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→ **Algorithm R*$_\delta$**:

- Besides an estimation function $h$ for $h^*$ there is also knowledge about the uncertainty of the estimation process.
- Knowledge about the uncertainty of the estimation process is expressed in the form of a probability density function $\rho_{h^*}(x)$. 


Using Information about Uncertainty of $h$

Describing the Estimation Uncertainty using Density Functions

Viewing cost functions as a random variables:

<table>
<thead>
<tr>
<th>cost function</th>
<th>random variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h^*(n)$</td>
<td>$h^*_n$</td>
</tr>
<tr>
<td>$f^<em>(n) = g^</em>(n) + h^*(n)$</td>
<td>$f^*_n$</td>
</tr>
<tr>
<td>$f^+(n) = g(n) + h^*(n)$</td>
<td>$f^+_n$</td>
</tr>
</tbody>
</table>

Let $\rho_{h^*_n}$ be a density function for the random variable $h^*_n$.

Semantics:

On the basis of $\rho_{h^*_n}$ one can define the probability with which $h^*(n)$ can be found in a neighborhood of $x$ costs.

\[ P(h^*_n = x) = \rho_{h^*_n}(x) \]
Using Information about Uncertainty of \( h \)
Describing the Estimation Uncertainty using Density Functions (continued)

Let \( \rho_{h_n^*} \) be a density function for the random variable \( h_n^* \).

Further applies:

1. From \( \rho_{h_n^*}(x) \) a density function \( \rho_{f_n^*}(y) \) can be derived for the random variable \( f_n^* \), if \( g^* \) is known (e.g., when searching a tree):

\[
\rho_{f_n^*}(y) := \rho_{h_n^*}(y - g^*)
\]

2. Let \( P_{s-n} \) be the cheapest known path from \( s \) to an OPEN node \( n \).

From \( \rho_{h_n^*}(x) \) a density function \( \rho_{f_n^+}(y) \) can be derived for the random variable \( f_n^+ \), which specifies the cost of an optimal solution path that continues \( P_{s-n} \):

\[
\rho_{f_n^+}(y) := \rho_{h_n^*}(y - g)
\]
Remarks:

- The random variable $f_n^+$ with associated density function $\rho_{f_n^+}$ is given for each node $n$.
- The random variable $f_n^+$ describes the possible costs of an optimal solution path that contains the pointer path $P_{s-n}$ as a subpath.
- If goal nodes can be reached from $s$, the OPEN list always contains a node $n$, to which $f^+(n) = f^*(n)$ applies. [Corollary Shallowest OPEN Node on Optimum Path]
Using Information about Uncertainty of $h$

Describing the Estimation Uncertainty using Density Functions

(continued)

Uncertainty area:

Density functions $\rho$:

Related distribution function:
Using Information about Uncertainty of $h$

Describing the Estimation Uncertainty using Density Functions (continued)

How should an evaluation order be calculated from the density functions $\rho_{f_n^+}$ for the nodes in the OPEN list?

Possible shapes of two density functions:

- **Case (a)**
- **Case (b)**
Remarks:

(a) If the density functions do not overlap, the node for which the corresponding density function
\( \rho f^+ \) has the lowest density value \( f^+_a \) with respect to all other nodes would be selected.

(b) \( f^+_{n_1} \) has the lower expected value; \( n_2 \) has the possibility that the cost \( f^+_{n_2} \) may be lower than \( n_1 \). An admissible algorithm would expand \( n_2 \).

It would make more sense to expand \( n_1 \) because the “\( f^+(n_2) < f^+(n_1) \)” event is unlikely.

\( \rightarrow \) Due to uncertainty, costs can be overestimated or underestimated. I.e., not expanding a node in OPEN and terminating it too expensively as a result, represents a risk.

\( \rightarrow \) Quantification of the risk of terminating with too high costs (= terminating too early).
Risk Measures
Defining the Order of Node Evaluations

Idea:
Estimate the risk of terminating too early using a risk measure $R$ for each node in the OPEN list.

- For a given cost value $C$ (of a goal node), the risk measure evaluates for each node $n$ in the OPEN list to what extent $C$ can be improved by expanding $n$.
- $R = R(C)$. The risk measure is a nondecreasing function of the $C$ cost. The greater the $R(C)$ value of a node $n$, the greater the risk of missing an improvement of $C$ if terminating with $C$ without expanding $n$.
- $R(C)$ should use knowledge about the cost distribution for the node $n$, so it should be based on $\rho f_n^+$. 
Risk Measures

Principle of the Algorithm $R^*_\delta$

Search continues until the risk value $R(C)$ of each node in the OPEN list is below a user accepted risk threshold $\delta$.

- If a high risk is acceptable, nodes with a high risk value of $R(C)$ (= high cost reduction potential) remain unexpanded. As a result, cost underestimation becomes less likely.
- If only a small risk is acceptable, even nodes with a low risk value of $R(C)$ (= low cost reduction potential) are expanded. As a result, cost underestimation becomes more likely.

- I.e., depending on a risk threshold $\delta$ the probability of a cost underestimation (= probability of admissibility or optimality) can be controlled.
Remarks:

- The observance of this principle by the algorithm $R^*_\delta$ is ensured — as shown later — by the use of certain risk functions $R(C)$.

- When a goal node with cost $C$ is selected from OPEN, its risk function must guarantee the property $C \leq C_\delta$. Otherwise, one of the remaining nodes in OPEN can have a risk for $C$ that is higher than $\delta$. 
Risk Measures
Potential for Improvement to a Current Solution

Let $n_1, n_2$ be nodes of the OPEN list.

Possible improvement by expanding $n_1$

Possible improvement by expanding $n_2$

Cost of a solution already found

Example of risk functions $R(C)$ for the nodes $n_1, n_2$:

The nodes have different random variables $f_{n_1}^+$ and $f_{n_2}^+$ for the cost.
Remarks:

- The potential for improvement is a statistical quantity defined for a node $n$ using $f_n^+$. 
- The *evaluation* of the potential for improvement regarding given costs $C'$ is done with the help of a risk-measure $R(C')$. 
Risk Measures
Risk Threshold and Cost Threshold

The risk threshold $\delta \geq 0$ defines for each node $n$ in OPEN its cost threshold $C_\delta(n)$:

Let $n_1, n_2$ be nodes in OPEN. If the search was terminated with node $n_2$ and cost $C'' = C_\delta(n_2)$, the risk $R(C'')$ for $n_1$ would be above the risk threshold $\delta$.

$\Rightarrow$ $R^*_\delta$ chooses the node $n$ with the lowest cost threshold $C_\delta(n)$ in the OPEN list. In the above example, node $n_1$ would be preferred to node $n_2$. 
Risk Measures

Definition 75 (Risk Measure)
Let $M$ be the ordered set of cost values. A risk measure $R(C)$ for a node is a nondecreasing function $R : M \rightarrow [0, \infty]$ measuring the risk associated with leaving that node unexplored when terminating with a solution with cost $C$.

Definition 76 (Cost Threshold)
Let $\delta$ be a nonnegative real number and let $R(C)$ be the risk measure for a node $n$. The solution $C_\delta(n)$ to the equation $R(C) = \delta$ is called the cost threshold.

Assuming the cost of a solution path found is $C$, then for each node $n$ in OPEN with $C > C_\delta(n)$ the risk of missing a better solution path is higher than risk threshold $\delta$. These nodes should be expanded before termination.
Remarks:

- Risk measures and risk thresholds must be seen in context: not every risk threshold makes sense for a risk measure.

- Depending on the $f^+_n$ cost random variable of a $n$ node, the $\delta$ risk threshold can lead to different sequences in the OPEN list.

- The cost-threshold $C_\delta(n)$ indicates how high the cost of a solution may be without exceeding the $\delta$-risk-threshold for the node $n$. 
Risk Measures

Definition 77 (δ-Risk-Admissibility)
An algorithm is said to be δ-risk-admissible if it always terminates with a solution cost \( C \) such that \( R(C) \leq \delta \) for each node left on OPEN.

The above version of the δ-risk-admissible condition is equivalent to stating that at termination, the cost of the solution found is not greater than \( C_\delta(n) \) for each \( n \) on OPEN.

Definition 78 (Algorithm R*δ)

R*δ is a search algorithm which is identical to A* except that it chooses for expansion that node \( n \) from OPEN with the lowest cost-threshold \( C_\delta(n) \).

Note that with \( \delta = 0 \), R*δ is identical to A* since it is guided by the (admissible) lowest tail of the density of \( f \), namely by \( g + h_a \).

For \( \delta > 0 \), R*δ may prefer a node with high \( f_a \) and narrow distribution over a node with low \( f_a \) but highly diffused density.
Remarks:

- The first definition of $\delta$-risk-admissibility is from the perspective of risk, the second is from the perspective of cost.
- With $\delta = 0$, $R^*_\delta$ is identical to $A^*$. Justification:
  1. Computing the cost-threshold $C_{\delta}(n)$ for nodes in OPEN is solving the equation $R(C) = 0$ for $\delta = 0$.
  2. $R(C) = 0$ holds for the lowest point $f_a$ on the tail of the density of $f^+$.
  3. Hence, $f_a = g + h_a \leq g(n) + h^*(n)$
  4. $R^*_\delta$ is guided by an admissible heuristic function and, therefore, $R^*_\delta$ is admissible.
- As the $\delta$ increases, $R^*_\delta$ tends to abandon admissibility.
Risk Measures
Risk Measures of Type \( R(C) = \varrho[C - f^+] \)

Starting point are density functions for the random variables \( f_n^+ \) of nodes \( n \) in the OPEN list.

Examples:

\[ f_a (\text{resp. } h_a) \text{ is the smallest positive preimage of the density function } \rho_{f_n^+} (\text{resp. } \rho_{h_n^*}). \]
Risk Measures
Risk Measures of Type $R(C) = \varrho[C - f^+]$ (continued)

1. Worst Case Risk $R_1$:

$$R_1(C) = \sup_{\{y | \rho_{f^+}(y) > 0\}} (C - y) = C - f_a = C - g - h_a$$

2. Probability of Suboptimal Termination $R_2$:

$$R_2(C) = P(C > f^+_n) = P(C - f^+_n > 0) = \int_{y=-\infty}^{C} \rho_{f^+_n}(y) \, dy$$

3. Expected Risk $R_3$:

$$R_3(C) = E(\max\{C - f^+_n; 0\}) = \int_{y=-\infty}^{C} (C - y) \rho_{f^+_n}(y) \, dy$$
Remarks:

- The risk measures $R_1$ and $R_3$ describe costs, the risk measure $R_2$ describes a probability.
  
  - $R_1$:
    For the costs represented by the $f_n^+$ random variable, the smallest possible value is assumed. $R_1$ quantifies the maximum possible loss if a solution is satisfied with $C$ costs. 
    The lowest costs are the worst case because they represent the extreme case of a missed cost reduction. The probability that the remaining costs are lower than $h_a$ is 0.

  - $R_2$:
    The probability for the occurrence of the event "$C > f_n^+$" (i.e., event is a loss) is calculated if you are satisfied with a solution with $C$ cost.

  - $R_3$:
    For the costs represented by the random variable $f_n^+$, the expected loss $E(\max\{C - f_n^+; 0\})$ is calculated if one is satisfied with a solution with costs $C$.
    $R_3$ weights the probability of the loss ($R_2$) with the amount of the occurring loss.
Risk Measures

Example

Let $f_n^+$ be uniformly distributed between an optimistic estimate $f_a$ and a pessimistic estimate $f_b$ (The $f_a$ and $f_b$ estimates depend on $n$, where $n$ is in OPEN.):

$$f_a = g + h_a \quad f_b = g + h_b$$

Density function:

$$\rho_{f_n^+}(y) = \begin{cases} 
\frac{1}{f_b - f_a} & f_a \leq y \leq f_b \\
0 & \text{else}
\end{cases}$$

Risk measure:

$$R_1(C) = \sup_{\{y | \rho_{f_n^+}(y) > 0\}} (C - y) = C - f_a$$
Let $f_n^+$ be uniformly distributed between an optimistic estimate $f_a$ and a pessimistic estimate $f_b$ (The $f_a$ and $f_b$ estimates depend on $n$, where $n$ is in OPEN.):

$$f_a = g + h_a \quad f_b = g + h_b$$

Density function:

$$\rho_{f_n^+}(y) = \begin{cases} \frac{1}{f_b - f_a} & f_a \leq y \leq f_b \\ 0 & \text{else} \end{cases}$$

Risk measure:

$$R_2(C) = \int_{y=-\infty}^{C} \rho_{f_n^+}(y) dy = \begin{cases} 0 & C < f_a \\ \frac{(C-f_a)}{(f_b-f_a)} & f_a \leq C \leq f_b \\ 1 & f_b < C \end{cases}$$
Risk Measures

Example (continued)

Let $f_n^+$ be uniformly distributed between an optimistic estimate $f_a$ and a pessimistic estimate $f_b$ (The $f_a$ and $f_b$ estimates depend on $n$, where $n$ is in OPEN.):

Density function:

$$\rho_{f_n^+}(y) = \begin{cases} 
\frac{1}{f_b - f_a} & f_a \leq y \leq f_b \\
0 & \text{else}
\end{cases}$$

Risk measure:

$$R_3(C) = \int_{y=-\infty}^{C} (C - y) \rho_{f_n^+}(y) dy = \begin{cases} 
0 & C < f_a \\
\frac{(C-f_a)^2}{2(f_b-f_a)} & f_a \leq C \leq f_b \\
C - \frac{f_a+f_b}{2} & f_b < C
\end{cases}$$
Risk Measures
Example  (continued)

Shape of risk measures $R_1$, $R_2$, and $R_3$ ($f_n^+$ uniformly distributed):

The vertical axis represents the functions $R(C)$ for the three risk measures considered.
Computing the cost threshold $C_\delta$ for $R_3 \left( f_n^+ \right)$ uniformly distributed:

Let $\delta$ be the user’s risk tolerance of the user. For each node $n$ in OPEN, it defines its cost threshold $C_\delta(n)$ using the equation $R(C) = \delta$.

For a node $n$ on OPEN, $C_\delta(n)$ is computed by transforming $R_3(C_\delta) = \delta$:

$$C_\delta(n) = \begin{cases} 
  f_a & \delta = 0 \\
  f_a + \sqrt{2 \cdot (f_b - f_a) \cdot \delta} & 0 < \delta \leq \frac{f_b - f_a}{2} \\
  \delta + \frac{f_a + f_b}{2} & \frac{f_b - f_a}{2} < \delta 
\end{cases}$$
Risk Measures

Example (continued)

For a goal node $\gamma$ in OPEN, we have $h(n) = h^*(n) = 0$. Therefore, there remains no uncertainty regarding $f^+_\gamma$.

Graphs of random variable $f^+_\gamma$ for solution cost and risk measures $R_1$, $R_2$, and $R_3$:

Cost threshold:

$$C_{\delta}(\gamma) = \begin{cases} 
  g(\gamma) + \delta & \text{for risk measure } R_1 \\
  g(\gamma) & \text{for risk measure } R_2 \text{ and } \delta < 1 \\
  g(\gamma) + \delta & \text{for risk measure } R_3
\end{cases}$$
Theorem 79 (δ-Risk-Admissibility of $R^*_\delta$)

$R^*_\delta$ is $\delta$-risk-admissible with respect to risk measures $R_1$, $R_2$, and $R_3$ when $G$ is a search space graph with $Prop(G)$ and $E(h^*) < \infty$ on solution paths.
Risk Measures

**Theorem 79 (δ-Risk-Admissibility of R\(^*\)\(_{\delta}\))**

R\(^*\)\(_{\delta}\) is δ-risk-admissible with respect to risk measures \(R_1\), \(R_2\), and \(R_3\) when \(G\) is a search space graph with \(Prop(G)\) and \(E(h^*) < \infty\) on solution paths.

**Proof (sketch)**

1. δ-Risk-Admissibility:
   1. (a) According to the previous example, it holds for the cost \(C\) of a solution path found by \(R^*_{\delta}\):
      \[
      C = g(\gamma) \leq C^\delta(\gamma)
      \]
      for the risk measures \(R_1\), \(R_2\), \(R_3\).
   1. (b) Since \(R^*_{\delta}\) chooses for expansion that node \(n\) from OPEN with the lowest cost-threshold \(C^\delta(n)\), δ-risk-admissibility of \(R^*_{\delta}\) follows for risk measures \(R_1\), \(R_2\), and \(R_3\).

2. Completeness:
   1. (a) At all times OPEN contains a node \(n\) on a solution path for which \(C^\delta(n)\) is finite. Obviously, \(R_1(C) = \delta\) and \(R_2(C) = \delta\) have a finite solution. If density \(\rho_{h^*}(x)\) possesses a finite expectation \(E(h^*) < \infty\) for any node on a solution path, for \(R_3\) we have
      \[
      R_3(C) \geq C \cdot (1 - P(f^+ > C)) = C - E(f^+) \geq C - 2E(f^+) = C - 2g - 2E(h^*)
      \]
   1. (b) \(C^\delta(n) \geq g(n)\) holds for each node \(n\) in OPEN since there is no risk in abandoning \(n\) after finding a solution path with cost \(\leq g(n)\). A positive lower bound of the edge cost values guarantees that \(R^*_{\delta}\) can neglect nodes on solution paths only for a limited number of node expansions.
Remarks:

- Expectations can have the value $\infty$, e.g., for a random variable that returns values $2^n$ with probability $2^{-n}$.

- In step 2(a) we use the fact $R_3(0) = 0$ for graphs $G$ with nonnegative edge cost values. As the lower bound for $R_3(C')$ increases with $C'$, there is a finite value $C$ with $R_3(C') > \delta$. Hence, $C_\delta(n) < \infty$.

- The exact form of $\rho h_n^*$ is generally unknown. For this the edge costs must have been generated by a given probabilistic model.

- Generating a good estimate for $C_\delta(n)$ is often possible. For this, the knowledge of upper and lower bounds of $h_n^*$ together with the often reasonable assumption of a standardized distribution between them, such as an uniform distribution, an exponential distribution or a normal distribution, is sufficient.

- The principle of the $\varepsilon$-admissible acceleration in $A^*_\varepsilon$ for $A^*$ can also be applied to $R^*_\delta$ and leads to the algorithm $R^*_{\delta,\varepsilon}$. The special version $R^*_{\delta,\delta}$ is $\delta$-risk-admissible with respect to risk measures $R_1$, $R_2$, and $R_3$ under the preconditions of the previous theorem.